5-2 Properties of Quadratic Functions in Standard Form

TEKS 2A.7.A Quadratic and square root functions: . . . connect between the $y = ax^2 + bx + c$ and the $y = a(x - h)^2 + k$ symbolic representations . . . .

Objectives

Define, identify, and graph quadratic functions.

Identify and use maximums and minimums of quadratic functions to solve problems.

Vocabulary

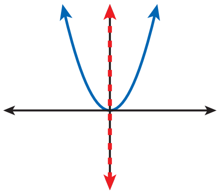
axis of symmetry
standard form
minimum value
maximum value

Why learn this?

Quadratic functions can be used to find the maximum power generated by the engine of a speedboat. (See Example 4.)

When you transformed quadratic functions in the previous lesson, you saw that reflecting the parent function across the y-axis results in the same function.

$f(x) = x^2$
$g(x) = (-x)^2 = x^2$

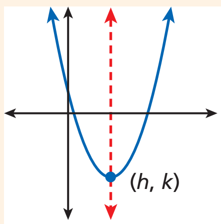


Also 2A.1.A, 2A.6.A, 2A.6.B, 2A.6.C, 2A.8.A

This shows that parabolas are symmetric curves. The **axis of symmetry** is the line through the vertex of a parabola that divides the parabola into two congruent halves.

Know it! Note

Axis of Symmetry — Quadratic Functions

WORDS	ALGEBRA	GRAPH
The axis of symmetry is a vertical line through the vertex of the function's graph.	The quadratic function $f(x) = a(x - h)^2 + k$ has the axis of symmetry $x = h$.	

EXAMPLE 1 **Identifying the Axis of Symmetry**

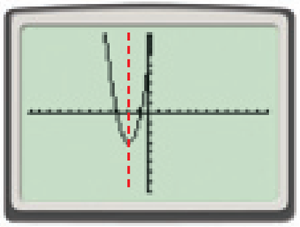
Identify the axis of symmetry for the graph of $f(x) = 2(x + 2)^2 - 3$.

Rewrite the function to find the value of h.

$$f(x) = 2[x - (-2)]^2 - 3$$

Because $h = -2$, the axis of symmetry is the vertical line $x = -2$.

Check Analyze the graph on a graphing calculator. The parabola is symmetric about the vertical line $x = -2$.



CHECK IT OUT!

1. Identify the axis of symmetry for the graph of $f(x) = (x - 3)^2 + 1$.

Another useful form of writing quadratic functions is the *standard form*. The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$.

The coefficients a, b, and c can show properties of the graph of the function. You can determine these properties by expanding the vertex form.

$$f(x) = a(x - h)^2 + k$$

$$f(x) = a(x^2 - 2xh + h^2) + k \qquad \textit{Multiply to expand } (x - h)^2.$$

$$f(x) = a(x^2) - a(2hx) + a(h^2) + k \qquad \textit{Distribute } a.$$

$$f(x) = ax^2 + (-2ah)x + (ah^2 + k) \qquad \textit{Simplify and group like terms.}$$

$$\overset{a = a}{\downarrow} \quad \overset{-2ah = b}{\downarrow} \quad \overset{ah^2 + k = c}{\downarrow}$$

$$f(x) = ax^2 + bx + c$$

$a = a$: a in standard form is the same as in vertex form. It indicates whether a reflection and/or vertical stretch or compression has been applied.

$b = -2ah$: Solving for h gives $h = \frac{b}{-2a} = -\frac{b}{2a}$. Therefore, the axis of symmetry, $x = h$, for a quadratic function in standard form is $x = -\frac{b}{2a}$.

$c = ah^2 + k$: Notice that the value of c is the same value given by the vertex form of f when $x = 0$: $f(0) = a(0 - h)^2 + k = ah^2 + k$. So c is the y-intercept.

These properties can be generalized to help you graph quadratic functions.

Know it! Note

Properties of a Parabola

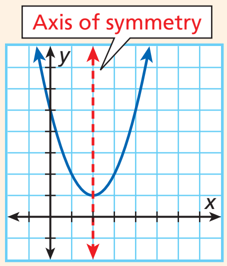
For $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, the parabola has these properties:

The parabola **opens** upward if $a > 0$ and downward if $a < 0$.

The **axis of symmetry** is the vertical line $x = -\frac{b}{2a}$.

The **vertex** is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

The ***y*-intercept** is c.



EXAMPLE 2 Graphing Quadratic Functions in Standard Form

A **Consider the function $f(x) = x^2 - 4x + 6$.**

a. Determine whether the graph opens upward or downward.

Because a is positive, the parabola opens upward.

b. Find the axis of symmetry.

The axis of symmetry is given by $x = -\frac{b}{2a}$.

$$x = -\frac{(-4)}{2(1)} = 2 \qquad \textit{Substitute } -4 \textit{ for } b \textit{ and } 1 \textit{ for } a.$$

The axis of symmetry is the line $x = 2$.

c. Find the vertex.

The vertex lies on the axis of symmetry, so the x-coordinate is 2. The y-coordinate is the value of the function at this x-value, or $f(2)$.

$$f(2) = (2)^2 - 4(2) + 6 = 2$$

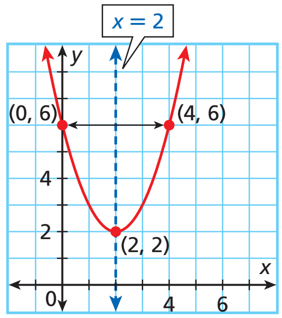
The vertex is $(2, 2)$.

d. Find the y-intercept.

Because $c = 6$, the y-intercept is 6.

e. Graph the function.

Graph by sketching the axis of symmetry and then plotting the vertex and the intercept point, $(0, 6)$. Use the axis of symmetry to find another point on the parabola. Notice that $(0, 6)$ is 2 units left of the axis of symmetry. The point on the parabola symmetrical to $(0, 6)$ is 2 units right of the axis at $(4, 6)$.



B **Consider the function $f(x) = -4x^2 - 12x - 3$.**

a. Determine whether the graph opens upward or downward.

Because a is negative, the parabola opens downward.

Helpful Hint

When a is positive, the parabola is happy (∪). When a is negative, the parabola is sad (∩).

b. Find the axis of symmetry.

The axis of symmetry is given by $x = -\frac{b}{2a}$.

$$x = -\frac{(-12)}{2(-4)} = -\frac{3}{2} \quad \text{Substitute } -12 \text{ for } b \text{ and } -4 \text{ for } a.$$

The axis of symmetry is the line $x = -\frac{3}{2}$, or $x = -1.5$.

c. Find the vertex.

The vertex lies on the axis of symmetry, so the x-coordinate is -1.5.

The y-coordinate is the value of the function at this x-value, or $f(-1.5)$.

$$f(-1.5) = -4(-1.5)^2 - 12(-1.5) - 3 = 6$$

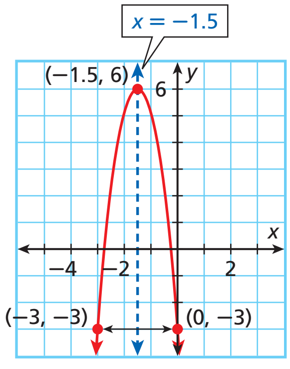
The vertex is $(-1.5, 6)$.

d. Find the y-intercept.

Because $c = -3$, the y-intercept is -3.

e. Graph the function.

Graph by sketching the axis of symmetry and then plotting the vertex and the intercept point, $(0, -3)$. Use the axis of symmetry to find another point on the parabola. Notice that $(0, -3)$ is 1.5 units right of the axis of symmetry. The point on the parabola symmetrical to $(0, -3)$ is 1.5 units left of the axis at $(-3, -3)$.



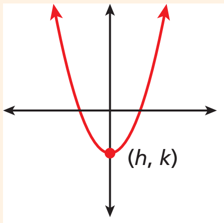
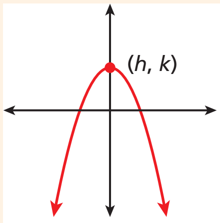
CHECK IT OUT! **For each function, (a) determine whether the graph opens upward or downward, (b) find the axis of symmetry, (c) find the vertex, (d) find the y-intercept, and (e) graph the function.**

2a. $f(x) = -2x^2 - 4x$ **2b.** $g(x) = x^2 + 3x - 1$

Substituting any real value of x into a quadratic equation results in a real number. Therefore, the domain of any quadratic function is all real numbers, $\mathbb{R}$. The range of a quadratic function depends on its vertex and the direction that the parabola opens.

Know it! Note

Minimum and Maximum Values

OPENS UPWARD	OPENS DOWNWARD
When a parabola opens upward, the y-value of the vertex is the **minimum value**.	When a parabola opens downward, the y-value of the vertex is the **maximum value**.
D: $\{x \mid x \in \mathbb{R}\}$ R: $\{y \mid y \geq k\}$ 	D: $\{x \mid x \in \mathbb{R}\}$ R: $\{y \mid y \leq k\}$ 
The domain is all real numbers, $\mathbb{R}$. The range is all values greater than or equal to the minimum.	The domain is all real numbers, $\mathbb{R}$. The range is all values less than or equal to the maximum.

EXAMPLE 3

Finding Minimum or Maximum Values

Find the minimum or maximum value of $f(x) = 2x^2 - 2x + 5$. Then state the domain and range of the function.

Step 1 Determine whether the function has a minimum or maximum value. Because a is positive, the graph opens upward and has a minimum value.

Step 2 Find the x-value of the vertex.

$$x = -\frac{b}{2a} = -\frac{(-2)}{2(2)} = \frac{2}{4} = \frac{1}{2}$$

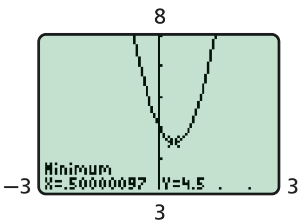
Substitute −2 for b and 2 for a.

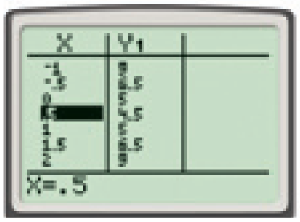
Step 3 Then find the y-value of the vertex, $f\left(-\frac{b}{2a}\right)$.

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 5 = 4\frac{1}{2}$$

The minimum value is $4\frac{1}{2}$, or 4.5. The domain is all real numbers, $\mathbb{R}$. The range is all real numbers greater than or equal to 4.5, or $\{y \mid y \geq 4.5\}$.

Check Graph $f(x) = 2x^2 - 2x + 5$ on a graphing calculator. The graph and table support the answer.





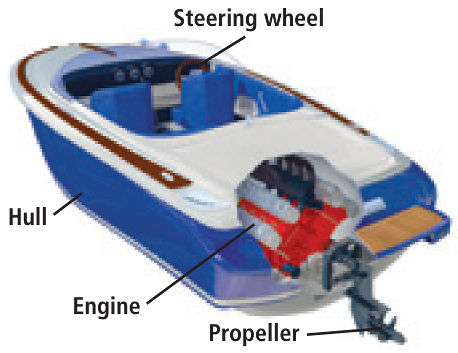
Caution!

The minimum (or maximum) value is the *y-value* of the vertex. It is *not* the ordered pair representing the vertex.

CHECK IT OUT! **Find the minimum or maximum value of each function. Then state the domain and range of the function.**

3a. $f(x) = x^2 - 6x + 3$ **3b.** $g(x) = -2x^2 - 4$

EXAMPLE 4 Transportation Application



The power p in horsepower (hp) generated by a high-performance speedboat engine operating at r revolutions per minute (rpm) can be modeled by the function $p(r) = -0.0000147r^2 + 0.18r - 251$. What is the maximum power of this engine to the nearest horsepower? At how many revolutions per minute must the engine be operating to achieve this power?

The maximum value will be at the vertex $\left(r, p(r)\right)$.

Step 1 Find the r-value of the vertex using $a = -0.0000147$ and $b = 0.18$.

$$r = -\frac{b}{2a} = -\frac{0.18}{2(-0.0000147)} \approx 6122$$

Step 2 Substitute this r-value into p to find the corresponding maximum, $p(r)$.

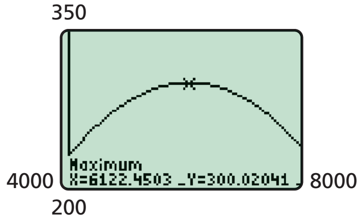
$$p(r) = -0.0000147r^2 + 0.18r - 251$$

$$p(6122) = -0.0000147(6122)^2 + 0.18(6122) - 251$$ *Substitute 6122 for r.*

$$p(6122) \approx 300$$ *Use a calculator.*

The maximum power is about 300 hp at 6122 rpm.

Check Graph the function on a graphing calculator. Use the **maximum** feature under the **CALCULATE** menu to approximate the maximum. The graph supports your answer.



CHECK IT OUT!

4. The highway mileage m in miles per gallon for a compact car is approximated by $m(s) = -0.025s^2 + 2.45s - 30$, where s is the speed in miles per hour. What is the maximum mileage for this compact car to the nearest tenth of a mile per gallon? What speed results in this mileage?

THINK AND DISCUSS

1. Explain whether a quadratic function can have both a maximum value and a minimum value.

2. Explain why the value of $f(x) = x^2 + 2x - 1$ increases as the value of x decreases from -1 to -10.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. In each box, write the criteria or equation to find each property of the parabola for $f(x) = ax^2 + bx + c$.